# SOLOMON'S INDUCTION IN QUASI-ELEMENTARY GROUPS

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Abstract. Given a finite group $G$, we address the following question: which multiples of the trivial representation are linear combinations of inductions of trivial representations from proper subgroups of $G$? By Solomon's induction theorem, all multiples are if $G$ is not quasi-elementary. We complement this by showing that all multiples of $p$ are if $G$ is $p$-quasi-elementary and not cyclic, and that this is best possible.

A finite group $G$ is ($p$-)*quasi-elementary* if it has a cyclic normal subgroup of $p$-power index. Solomon's induction theorem ([2] Thm. 1 with $\mathbf{K}=\mathbb{Q}$ or [1] Thm. 8.10) asserts that the trivial character of any finite group $G$ is a linear combination of inductions

$$\mathbf{1}_G = \sum_H n_H \operatorname{Ind}_H^G \mathbf{1}_H,$$

for some $n_H \in \mathbb{Z}$ and quasi-elementary subgroups $H<G$ (possibly with respect to different primes.) In particular, $\mathbf{1}_G$ is a linear combination of inductions of $\mathbf{1}_H$ from proper subgroups $H<G$ when $G$ is not quasi-elementary. In this note we show that this statement is never true for $\mathbf{1}_G$ when $G$ is $p$-quasi-elementary, but is always true for $p\mathbf{1}_G$, unless $G$ is cyclic. Both claims are easy to prove, but they do not appear to be in print.

We call a formal linear combination $\sum_H n_H H$ of (not necessarily proper) subgroups of $G$ a *Brauer relation* in $G$ if $\sum_H n_H \operatorname{Ind}_H^G \mathbf{1}_H = 0$.

**Theorem 1.** Let $G$ be a finite group, and let $I \subset \mathbb{Z}$ be the set of integers that can occur as $n_G$ in Brauer relations $\sum_H n_H H$. Then

- $I=\{0\}$ if $G$ is cyclic,
- $I=p\mathbb{Z}$ if $G$ is $p$-quasi-elementary and not cyclic, and
- $I=\mathbb{Z}$ if $G$ is not quasi-elementary.

*Proof.* Clearly $I$ is an ideal in $\mathbb{Z}$. It is easy to see that cyclic groups have no non-zero Brauer relations, whence the first claim. For the last claim, Solomon's induction theorem shows that $1 \in I$ for non-quasi-elementary groups. Assume from now on that $G$ is $p$-quasi-elementary and not cyclic. It remains to show that

a) $p\,\mathbf{1}_G$ is in $\mathbb{Z}$-span of $\operatorname{Ind}_H^G \mathbf{1}_H$ for $H \lneq G$, and
b) $\mathbf{1}_G$ is not.

Let $C \triangleleft G$ be a cyclic subgroup of $p$-power index. The elements of $C$ of order prime to $p$ form a cyclic subgroup $C'$ which is characteristic in $C$ and therefore normal in $G$. Replacing $C$ by $C'$, we may assume that $p \nmid |C|$. Now $G = C \rtimes P$ by the Schur-Zassenhaus theorem, with $P<G$ its $p$-Sylow.

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a) We proceed by induction on $|G|$.

If $N \triangleleft G$ is a normal subgroup and $\phi : G \twoheadrightarrow Q = G/N$ the quotient map, then any Brauer relation $\sum_U n_U U \ \ (U < Q)$ in $Q$ lifts to a relation $\sum_U n_U \phi^{-1}(U)$ in $G$. Also note that $Q$ is $p$-quasi-elementary as well. Thus if there exists an $N$ with $G/N$ *non-cyclic*, we may apply the theorem to $G/N$ (by induction) and lift the resulting relation back to $G$. Hence assume that there is no such $N$. This implies that

- *$P$ is cyclic.* Otherwise, let $N = C \rtimes$ (Frattini subgroup of $P$). Then $G/N \cong (C_p)^n$ for some $n > 1$, which is not cyclic.
- *The action of $P$ on $C$ is non-trivial.* Otherwise $G$ is cyclic.
- *The action of $P$ on $C$ is faithful.* Otherwise $G$ modulo the kernel of this action is a non-cyclic quotient.
- *$C$ has prime power order.* Otherwise $C = U_1 \times U_2$ with non-trivial $U_1$, $U_2$, and either $G/U_1$ or $G/U_2$ is a non-cyclic quotient.

In particular, because $P$ and $C$ have coprime order and $P \hookrightarrow \operatorname{Aut} C$, the order of $C$ cannot be a power of 2.

- *$C$ has prime order.* Otherwise take $U = C_{l^{k-1}} < C_{l^k} = C$. Then
$$(\mathbb{Z}/l\mathbb{Z})^\times \times (\mathbb{Z}/l^{k-1}\mathbb{Z}) \cong \operatorname{Aut}(C) \to \operatorname{Aut}(C/U) \cong (\mathbb{Z}/l\mathbb{Z})^\times$$
is bijective on elements of order prime to $l$, so $G$ acts faithfully on $C/U$, and $G/U$ is a non-cyclic quotient.

Finally, now $G = C_l \rtimes C_{p^k}$ with faithful action, and it is easy to check that
$$C_{p^{k-1}} - p\, C_{p^k} - C_l \rtimes C_{p^{k-1}} + p\, G = 0$$
is a Brauer relation.

b) Let $R = \sum n_H H$ be a Brauer relation. Restricting each term $\operatorname{Ind}_H^G \mathbf{1}_H$ to $C$ using Mackey's decomposition, we find a Brauer relation in $C$, namely
$$\sum n_H [G : HC]\ (H \cap C).$$
As cyclic groups have no non-trivial relations, all terms, in particular the ones with $C$ must cancel. These come from subgroups $H \supset C$, that is the ones of the form $H = C \rtimes P_H$ with $P_H \subset P$. Thus,
$$\sum_{H \supset C} n_H [P : P_H] = 0.$$

All terms except the one with $P_H = P$ (i.e. $H = G$) are divisible by $p$, so $n_G$ must be a multiple of $p$. This shows that $n_G \neq 1$. □

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## References

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